

Multiple Stirling Number Identities

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Abstract

A remarkable multiple analogue of the Stirling numbers of the first and second kind was recently constructed by the author. Certain summation identities, and related properties of this family of multiple special numbers are investigated in the present paper.

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1. Introduction

Stirling numbers of the first and second kind are studied and their properties are investigated extensively in number theory and combinatorics. One dimensional generalizations of these numbers have also been subject of interest. An important class of generalizations is their one parameter q -extensions. Many have made significant contributions to such q -extensions investigating their properties and applications. We will give references to some of these important work in Section 3 below.

In a recent paper, the author took a major step and constructed an elegant multiple qt -generalization of Stirling numbers of the first and second kind, besides sequences of other special numbers including multiple binomial, Fibonacci, Bernoulli, Catalan, and Bell numbers [16]. In this paper, we focus on multiple Stirling numbers of both kinds, and give interesting new identities satisfied by them.

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The multiple generalizations developed in [16] are given in terms of the qt -binomial coefficients constructed in the same paper. Its definition may be written in the general form as

$$\binom{z}{\mu}_{q,t} := \frac{q^{|\mu|} t^{2n(\mu) + (1-n)|\mu|}}{(qt^{n-1})_\mu} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j}} \right\} w_\mu(q^z t^{\delta(n)}; q, t)$$

where μ is a partition of at most n parts, $z \in \mathbb{C}^n$ and $q, t \in \mathbb{C}$. The w_μ function that enters the definition is a limiting case of the BC_n well-poised symmetric rational Macdonald function W_λ . Note that this definition makes sense even when μ is not an integer partition, but is a vector $\mu \in \mathbb{C}^n$.

2. Background

The basic q -Pochhammer symbol $(a; q)_\alpha$ may be defined formally for complex parameters $q, \alpha \in \mathbb{C}$ as

$$(a)_\alpha = (a; q)_\alpha := \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty} \quad (1)$$

where the infinite product $(a; q)_\infty$ is defined by $(a; q)_\infty := \prod_{i=0}^{\infty} (1 - aq^i)$. Note that when $\alpha = m$ is a positive integer, the definition reduces to the finite product $(a; q)_m = \prod_{k=0}^{m-1} (1 - aq^k)$. An elliptic analogue is defined [17, 36] by

$$(a; q, p)_m := \prod_{k=0}^{m-1} \theta(aq^m) \quad (2)$$

where $a \in \mathbb{C}$, m is a positive integer, and the normalized elliptic function $\theta(x)$ is given by

$$\theta(x) = \theta(x; p) := (x; p)_\infty (p/x; p)_\infty \quad (3)$$

for $x, p \in \mathbb{C}$ with $|p| < 1$. The definition is extended to negative m by setting $(a; q, p)_m = 1/(aq^m; q, p)_{-m}$. It is clear that when $p = 0$, the elliptic $(a; q, p)_m$ reduces to the basic (trigonometric) q -Pochhammer symbol (1).

For any partition $\lambda = (\lambda_1, \dots, \lambda_n)$ and $t \in \mathbb{C}$, define [39]

$$(a)_\lambda = (a; q, p, t)_\lambda := \prod_{k=1}^n (at^{1-i}; q, p)_{\lambda_i}. \quad (4)$$

Note that when $\lambda = (\lambda_1) = \lambda_1$ is a single part partition, then $(a; q, p, t)_\lambda = (a; q, p)_{\lambda_1} = (a)_{\lambda_1}$. For brevity of notation, we also use

$$(a_1, \dots, a_k)_\lambda = (a_1, \dots, a_k; q, p, t)_\lambda := (a_1)_\lambda \dots (a_k)_\lambda. \quad (5)$$

Recall that we use V to denote [15] the space of infinite lower-triangular matrices whose entries are rational functions over the field $\mathbb{F} = \mathbb{C}(q, p, t, r, a, b)$ which are indexed by partitions with respect to the partial inclusion ordering \subseteq defined by

$$\mu \subseteq \lambda \Leftrightarrow \mu_i \leq \lambda_i, \quad \forall i \geq 1. \quad (6)$$

The condition that a matrix $u \in V$ is lower triangular with respect to \subseteq can be stated in the form

$$u_{\lambda\mu} = 0, \quad \text{when } \mu \not\subseteq \lambda. \quad (7)$$

The multiplication operation defined by

$$(uv)_{\lambda\mu} := \sum_{\mu \subseteq \nu \subseteq \lambda} u_{\lambda\nu} v_{\nu\mu} \quad (8)$$

for matrices $u, v \in V$ makes V into an algebra over \mathbb{F} .

2.1. Well-poised Macdonald functions

The construction of our multiple Stirling numbers involves the elliptic well-poised Macdonald functions $W_{\lambda/\mu}$ on BC_n [15]. These remarkable families of symmetric rational functions are first introduced in the author's Ph.D. thesis [13] in the basic (trigonometric) case, and later in [14] in the more general elliptic form.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ be partitions of at most n parts for a positive integer n such that the skew partition λ/μ is a horizontal strip; i.e. $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \lambda_n \geq \mu_n \geq \lambda_{n+1} = \mu_{n+1} = 0$. Following [15], define

$$H_{\lambda/\mu}(q, p, t, b) := \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{\mu_i - \mu_{j-1}} t^{j-i})_{\mu_{j-1} - \lambda_j} (q^{\lambda_i + \lambda_j} t^{3-j-i} b)_{\mu_{j-1} - \lambda_j}}{(q^{\mu_i - \mu_{j-1} + 1} t^{j-i-1})_{\mu_{j-1} - \lambda_j} (q^{\lambda_i + \lambda_j + 1} t^{2-j-i} b)_{\mu_{j-1} - \lambda_j}} \cdot \frac{(q^{\lambda_i - \mu_{j-1} + 1} t^{j-i-1})_{\mu_{j-1} - \lambda_j}}{(q^{\lambda_i - \mu_{j-1}} t^{j-i})_{\mu_{j-1} - \lambda_j}} \right\} \cdot \prod_{1 \leq i < (j-1) \leq n} \frac{(q^{\mu_i + \lambda_j + 1} t^{1-j-i} b)_{\mu_{j-1} - \lambda_j}}{(q^{\mu_i + \lambda_j} t^{2-j-i} b)_{\mu_{j-1} - \lambda_j}} \quad (9)$$

and

$$W_{\lambda/\mu}(x; q, p, t, a, b) := H_{\lambda/\mu}(q, p, t, b) \cdot \frac{(x^{-1}, ax)_\lambda (qbx/t, qb/(axt))_\mu}{(x^{-1}, ax)_\mu (qbx, qb/(ax))_\lambda} \cdot \prod_{i=1}^n \left\{ \frac{\theta(bt^{1-2i} q^{2\mu_i})}{\theta(bt^{1-2i})} \frac{(bt^{1-2i})_{\mu_i + \lambda_{i+1}}}{(bqt^{-2i})_{\mu_i + \lambda_{i+1}}} \cdot t^{i(\mu_i - \lambda_{i+1})} \right\} \quad (10)$$

where $q, p, t, x, a, b \in \mathbb{C}$. Note that $W_{\lambda/\mu}(x; q, p, t, a, b)$ vanishes unless λ/μ is a horizontal strip. The function $W_{\lambda/\mu}(y, z_1, \dots, z_\ell; q, p, t, a, b)$ is extended to $\ell + 1$ variables $y, z_1, \dots, z_\ell \in \mathbb{C}$ through the following recursion formula

$$W_{\lambda/\mu}(y, z_1, z_2, \dots, z_\ell; q, p, t, a, b) = \sum_{\nu \prec \lambda} W_{\lambda/\nu}(yt^{-\ell}; q, p, t, at^{2\ell}, bt^\ell) W_{\nu/\mu}(z_1, \dots, z_\ell; q, p, t, a, b). \quad (11)$$

2.2. The Limiting Cases

The Macdonald functions W_λ are essentially equivalent to BC_n abelian functions constructed independently in [35]. The limiting cases defined above are closely related to the Macdonald polynomials [31], and interpolation Macdonald polynomials [34].

The following limiting case of the basic (the $p = 0$ case of the elliptic) W functions will be used in our constructions below. The existence of these limits can be seen from ($p = 0$ case of) the definition (10), the recursion formula (11) and the limit rule

$$\lim_{a \rightarrow 0} a^{|\mu|} (x/a)_\mu = (-1)^{|\mu|} x^{|\mu|} t^{-n(\mu)} q^{n(\mu')} \quad (12)$$

where $|\mu| = \sum_{i=1}^n \mu_i$ and $n(\mu) = \sum_{i=1}^n (i-1)\mu_i$, and $n(\mu') = \sum_{i=1}^n \binom{\mu_i}{2}$. Denote $H_{\lambda/\mu}(q, t, b) = H_{\lambda/\mu}(q, 0, t, b)$, and for $x \in \mathbb{C}$ define

$$w_{\lambda/\mu}(x; q, t) := \lim_{s \rightarrow \infty} \left(s^{|\lambda| - |\mu|} \lim_{a \rightarrow 0} W_{\lambda/\mu}(x; q, t, a, as) \right) = (-q/x)^{-|\lambda| + |\mu|} q^{-n(\lambda') + n(\mu')} H_{\lambda/\mu}(q, t) \frac{(x^{-1})_\lambda}{(x^{-1})_\mu} \quad (13)$$

The recurrence formula for $w_{\lambda/\mu}$ function turns out to be

$$w_{\lambda/\mu}(y, z; q, t) = \sum_{\nu \prec \lambda} t^{\ell(|\lambda| - |\nu|)} w_{\lambda/\nu}(yt^{-\ell}; q, t) w_{\nu/\mu}(z; q, t) \quad (14)$$

Similarly, for $x \in \mathbb{C}$ define the dual function

$$\begin{aligned}\hat{w}_{\lambda/\mu}(x; q, t) &:= \lim_{s \rightarrow 0} \left(\lim_{a \rightarrow 0} W_{\lambda/\mu}(x; q, t, a, as) \right) \\ &= t^{-n(\lambda) + |\mu| + n(\mu)} H_{\lambda/\mu}(q, t) \frac{(x^{-1})_\lambda}{(x^{-1})_\mu}\end{aligned}\tag{15}$$

The recurrence formula for the dual $\hat{w}_{\lambda/\mu}(x; q, t)$ may be written as

$$\hat{w}_{\lambda/\mu}(y, z; q, t) = \sum_{\nu \prec \lambda} \hat{w}_{\lambda/\nu}(yt^{-\ell}; q, t) \hat{w}_{\lambda/\mu}(z; q, t)\tag{16}$$

for $y \in \mathbb{C}$ and $z \in \mathbb{C}^\ell$. We now recall some old, and derive some new basic properties of the w function and its dual, and their connections.

Corollary 1. *Let μ be an n -part partition, and $x = (x_1, \dots, x_n) \in \mathbb{C}^n$.*

(1) The w_μ and its dual \hat{w}_μ are flipped versions of one another. That is,

$$\hat{w}_\mu(x, q, t) = q^{-|\mu|} t^{-2n(\mu) - (n-1)|\mu|} w_\mu(1/x, 1/q, 1/t)$$

(2) The limit $\lim_{q \rightarrow 1} w_\mu(xt^{\delta(n)}; q, t)$ exists. For the particular case when $x = \lambda$ is a partition, we use the notation

$$\bar{w}_\mu(q^\lambda t^{\delta(n)}; 1, t) := \lim_{q \rightarrow 1} (1 - q)^{-\mu_1} w_\mu(q^\lambda t^{\delta(n)}; q, t)\tag{17}$$

PROOF. Both properties follow, by direct calculation, from the definition (13) of $w_{\lambda/\mu}$, the recurrence relation (14) for $w_{\lambda/\mu}$, and limit formula (12), and the flip formula

$$x^{|\mu|} (x^{-1}, q, t)_\mu = (-1)^{|\mu|} q^{n(\mu')} t^{-n(\mu)} (x; q^{-1}, t^{-1})_\mu\tag{18}$$

The proof also uses the result that, in the limit

$$\lim_{q \rightarrow 1} H_{\lambda/\mu}(t, q) = \frac{(k)_m}{m!}\tag{19}$$

where k is the maximum of the list $k = \max\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \dots, \lambda_{n-1} - \lambda_n\}$, and m is the maximum of $m = \max\{\lambda_1 - \mu_1, \mu_1 - \lambda_2, \lambda_2 - \mu_2, \dots, \mu_{n-1} - \lambda_n\}$. If the second list has a negative number (which means that λ/μ is not a horizontal strip), then $H_{\lambda/\mu}(t, q) = 0$.

Remark 1. We will need the following properties from [16] in what follows.

(3) If $z = xt^{\delta(n)}$, for some $x \in \mathbb{C}$, we get

$$w_\mu(xt^{\delta(n)}; q, t) = (-1)^{|\mu|} x^{|\mu|} t^{n(\mu)} q^{-|\mu| - n(\mu')} (x^{-1})_\mu \prod_{1 \leq i < j \leq n} \frac{(t^{j-i+1})_{\mu_i - \mu_j}}{(t^{j-i})_{\mu_i - \mu_j}} \quad (20)$$

which, after flipping q and t and using the flip rule (18), may be written as

$$w_\mu(xt^{\delta(n)}; q, t) = q^{-|\mu|} (x; q^{-1}, t^{-1})_\mu \prod_{1 \leq i < j \leq n} \frac{(t^{j-i+1})_{\mu_i - \mu_j}}{(t^{j-i})_{\mu_i - \mu_j}} \quad (21)$$

(4) The vanishing property of W functions implies that

$$w_\mu(q^\lambda t^\delta; q, t) = 0 \quad (22)$$

when $\mu \not\subseteq \lambda$, where \subseteq denotes the partial inclusion ordering.

(5) Let λ be an n -part partition with $\lambda_n \neq 0$ and $0 \leq k \leq \lambda_n$ for some integer k , and let $x = (x_1, \dots, x_n) \in \mathbb{C}^n$. It was shown in [16] that

$$w_{\bar{k}}(x; q, t) = q^{-nk} \prod_{i=1}^n (q^{1-k} x_i)_k \quad (23)$$

where $\bar{k} = (k, \dots, k) \in \mathbb{C}^n$.

2.3. qt -Binomial Coefficients

The multiple Stirling numbers we develop in this paper are closely connected with binomial coefficients just as in the one dimensional case. Recall that qt -binomial coefficient is defined by [16]

Definition 1. Let $z = (x_1, \dots, x_n) \in \mathbb{C}^n$ and μ be an n -part partition. Then the qt -binomial coefficient is defined by

$$\binom{z}{\mu}_{q,t} := \frac{q^{|\mu|} t^{2n(\mu) + (1-n)|\mu|}}{(qt^{n-1})_\mu} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j}} \right\} w_\mu(q^z t^{\delta(n)}; q, t) \quad (24)$$

where $q, t \in \mathbb{C}$. It should be noted that this definition makes sense even for $\mu \in \mathbb{C}^n$.

Setting $t = q^\alpha$ and sending $q \rightarrow 1$ yields the multiple ordinary α -binomial coefficients. For $n = 1$, the definition reduces to that of the one dimensional q -binomial coefficients

$$\binom{n}{k}_q := \frac{(q)_n}{(q)_{n-k}(q)_k} \quad (25)$$

which are also known as the Gaussian polynomials. These are studied extensively in the literature including but not limited to the works in [2, 3, 23, 4, 25, 32, 24, 12].

For the special case $z = \bar{x} = (x, \dots, x) \in \mathbb{C}^n$, we get

$$\begin{aligned} \binom{\bar{x}}{\mu}_{q,t} &= \frac{t^{2n(\mu)+(1-n)|\mu|}}{(qt^{n-1})_\mu} (q^x; q^{-1}, t^{-1})_\mu \\ &\quad \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j}} \frac{(t^{j-i+1})_{\mu_i - \mu_j}}{(t^{j-i})_{\mu_i - \mu_j}} \right\} \end{aligned} \quad (26)$$

Note that with this definition we can write the terminating qt -binomial theorem [16] in the form

$$(x)_\lambda = \sum_{\mu \subseteq \lambda} (-1)^{|\mu|} q^{n(\mu')} t^{-n(\mu)} \binom{\lambda}{\mu}_{q,t} x^{|\mu|} \quad (27)$$

We now recall another important extension [16] that generalize the one dimensional q -bracket to multiple qt -bracket as follows.

Definition 2. Let $z = (x_1, \dots, x_n) \in \mathbb{C}^n$. Then

$$\begin{aligned} [z, s]_\mu &:= [z, s, n, q, t]_\mu \\ &= q^{|\mu|} \prod_{i=1}^n \left\{ \frac{1}{(1 - qt^{n-i})^{\mu_i}} \right\} \prod_{1 \leq i < j \leq n} \left\{ \frac{(t^{j-i})_{\mu_i - \mu_j}}{(t^{j-i+1})_{\mu_i - \mu_j}} \right\} w_\mu(sq^z t^{\delta(n)}; q, t) \end{aligned} \quad (28)$$

is called the partition μ shifted qt -number or the qt -bracket. Note that the definition combines a multiplicative variable s , and an exponential variable z . Depending on the application we often set $z = \bar{0}$ and write $\langle s \rangle_\mu = [\bar{0}, s]_\mu$, or set $s = \bar{1}$ and write $[z]_\mu = [z, \bar{1}]_\mu$. In the special case when $\mu = \bar{1}$, we get

$$[z] = [z, \bar{1}, n, q, t]_{\bar{1}} = \binom{z}{\bar{1}}_{q,t} = \prod_{i=1}^n \frac{(1 - q^{x_i} t^{n-i})}{(1 - qt^{n-i})} \quad (29)$$

which is a multiple analogue of the classical q -bracket.

The definition $[z]_\mu$ may also be written as

$$[z]_\mu = t^{-2n(\mu)-(1-n)|\mu|} \prod_{i=1}^n \left\{ \frac{(qt^{n-i})_{\mu_i}}{(1-qt^{n-i})_{\mu_i}} \right\} \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(t^{j-i})_{\mu_i-\mu_j}}{(t^{j-i+1})_{\mu_i-\mu_j}} \frac{(qt^{j-i-1})_{\mu_i-\mu_j}}{(qt^{j-i})_{\mu_i-\mu_j}} \right\} \left(\frac{z}{\mu} \right)_{q,t} \quad (30)$$

which reduces to (28) when $\mu = \bar{1}$.

Note also that when $z = (x, \dots, x) = \bar{x} \in \mathbb{C}^n$ for a single variable $x \in \mathbb{C}$, the μ -shifted qt -number $[\bar{x}]_\mu$ may be written as

$$[\bar{x}]_\mu = \prod_{i=1}^n \left\{ \frac{1}{(1-qt^{n-i})_{\mu_i}} \right\} (q^x; q^{-1}, t^{-1})_\mu = \prod_{i=1}^n \left\{ \frac{(q^x t^{i-1}; q^{-1})_{\mu_i}}{(1-qt^{n-i})_{\mu_i}} \right\} \quad (31)$$

This definition reduces to the classical q -bracket in the one variable case.

Note that $(x; 1/q, 1/t)_\mu$, with the reciprocals of q and t , corresponds to a multiple basic qt -analogue of the falling factorial $x_{\underline{n}} := x(x-1) \cdots (x-(n-1))$ as opposed to the rising factorial or the Pochhammer symbol.

3. Multiple basic and ordinary qt -Stirling numbers

In this section we review the definition and fundamental properties of the multiple Stirling numbers of the first and second kind indexed by partitions [16]. The classical Stirling numbers of the first kind are defined to be the coefficients of the falling factorial $x_{\underline{n}}$ in the expansion

$$x_{\underline{n}} = n! \binom{x}{n} = \sum_{k=0}^n s_1(n, k) x^k \quad (32)$$

The q -analogue of these numbers are defined in [5] and their properties are studies in [33, 26, 29, 41] and the works of others.

First, we recall the definition of the multiple qt -Stirling numbers generalizing the one dimensional q -analogues.

Definition 3. For an n -part partition λ , the qt -Stirling numbers of the first kind $s_1(\lambda, \mu)$ are defined by

$$[x]_\lambda = \sum_{\mu \subseteq \lambda} q^{-n(\lambda')} t^{2n(\mu)-(n-1)|\mu|} s_1(\lambda, \mu) \prod_{i=1}^{\mu_1} [\bar{x}^i] \quad (33)$$

where $x \in \mathbb{C}$, and $\bar{x}^i = \{x, \dots, x, q, \dots, q\} \in \mathbb{C}^n$ with μ'_i copies of x for the dual partition μ' . That is,

$$\prod_{i=1}^{\mu_1} [\bar{x}^i] = \prod_{i=1}^n \frac{(1 - q^x t^{n-i})^{\mu_i}}{(1 - qt^{n-i})^{\mu_i}} \quad (34)$$

Likewise, the qt -Stirling numbers of the second kind $s_2(\lambda, \mu)$ are defined by

$$\prod_{i=1}^n \frac{(1 - q^x t^{n-i})^{\lambda_i}}{(1 - qt^{n-i})^{\lambda_i}} = \prod_{i=1}^{\lambda_1} [\bar{x}^i] = \sum_{\mu \subseteq \lambda} q^{n(\mu')} t^{-2n(\nu) + (n-1)|\nu|} s_2(\lambda, \mu) [x]_{\mu} \quad (35)$$

We now revise an explicit formula for the qt -Stirling numbers given in [16], generalizing one dimensional q -analogues as follows.

Theorem 2. For n -part partitions ν and μ , an explicit formula for the qt -Stirling numbers of first and second kind $s_1(\lambda, \mu)$ and $s_2(\lambda, \mu)$ are given by

$$s_1(\nu, \mu) = s_1(\nu, \mu, q, t) = \frac{q^{n(\nu')} t^{-2n(\mu) + 2(n-1)|\mu|}}{\prod_{i=1}^n (1 - qt^{n-i})^{\nu_i - \mu_i}} f_{\mu}(q, t) \cdot \sum_{\mu \subseteq \lambda \subseteq \nu} u(\nu, \lambda) t^{(1-n)|\lambda|} \bar{w}_{\mu}(q^{-\lambda} t^{-\delta(n)}; 1, 1/t) \quad (36)$$

and

$$s_2(\nu, \mu) = s_2(\nu, \mu, q, t) := \frac{q^{-n(\mu')} t^{2n(\nu) + (1-n)|\nu|}}{\prod_{i=1}^n (1 - qt^{n-i})^{\nu_i - \mu_i}} \cdot \sum_{\mu \subseteq \lambda \subseteq \nu} (-1)^{|\lambda|} t^{n(\lambda)} \bar{w}_{\lambda}(q^{\nu} t^{\delta(n)}; 1, t) f_{\lambda}(q, t) v(\lambda, \mu, q, t) \quad (37)$$

where \hat{w}_{μ} and \bar{w}_{μ} are as defined above in Corollary 1, and $f(\mu)$, $u(\lambda, \mu)$ and $v(\lambda, \mu)$ are given by

$$f(\mu, q, t) := \prod_{i=1}^{n-1} \frac{(t)_{\mu_i - \mu_{i+1}}}{(t^{n-i})_{\mu_i}} \prod_{\substack{1 \leq i < j \leq n \\ j \neq i+1}} \left\{ \frac{(t^{j-i})_{\mu_i - \mu_j}}{(t^{j-i-1})_{\mu_i - \mu_j}} \right\} \mu_n! \prod_{i=1}^{n-1} (\mu_i - \mu_{i+1})! \quad (38)$$

$$u(\lambda, \mu, q, t) := \frac{q^{|\mu|} t^{2n(\mu)}}{(qt^{n-1})_{\mu}} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j}} \right\} \hat{w}_{\mu}(q^{\lambda} t^{\delta(n)}; q, t) \quad (39)$$

and

$$v(\lambda, \mu, q, t) := \frac{(-1)^{|\mu|} q^{|\mu|+n(\mu')} t^{n(\mu)+(1-n)|\mu|}}{(qt^{n-1})_\mu} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j}} \right\} \\ \cdot w_\mu(q^\lambda t^{\delta(n)}; q, t) = (-1)^{|\mu|} q^{n(\mu')} t^{-n(\mu)} \binom{\lambda}{\mu}_{q,t} \quad (40)$$

PROOF. We write the W -Jackson sum [15] in the form

$$W_\lambda(x; q, p, t, at^{-2n}, bt^{-n}) \\ = \frac{(s)_\lambda (as^{-1}t^{-n-1})_\lambda}{(qbs^{-1}t^{-1})_\lambda (qbt^n s/a)_\lambda} \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(t^{j-i+1})_{\lambda_i - \lambda_j} (qbt^{-i-j+1})_{\lambda_i + \lambda_j}}{(t^{j-i})_{\lambda_i - \lambda_j} (qbt^{-i-j})_{\lambda_i + \lambda_j}} \right\} \\ \cdot \sum_{\mu \subseteq \lambda} \frac{(bs^{-1}t^{-n})_\mu (qbt^n/a)_\mu}{(qt^{n-1})_\mu (as^{-1}t^{-n-1})_\mu} \cdot \prod_{i=1}^n \left\{ \frac{(1 - bs^{-1}t^{1-2i} q^{2\mu_i})}{(1 - bs^{-1}t^{1-2i})} (qt^{2i-2})^{\mu_i} \right\} \\ \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(t^{j-i})_{\mu_i - \mu_j} (qt^{j-i})_{\mu_i - \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j} (t^{j-i+1})_{\mu_i - \mu_j}} \frac{(bs^{-1}qt^{-i-j})_{\mu_i + \mu_j} (bs^{-1}t^{-i-j+2})_{\mu_i + \mu_j}}{(bs^{-1}t^{-i-j+1})_{\mu_i + \mu_j} (qbs^{-1}t^{-i-j+1})_{\mu_i + \mu_j}} \right\} \\ \cdot W_\mu(q^\lambda t^{\delta(n)}; q, t, bt^{1-2n}, bs^{-1}t^{-n}) \cdot W_\mu(xs; q, t, as^{-2}t^{-2n}, bs^{-1}t^{-n}) \quad (41)$$

Set $b = ar$ in this identity and send $a \rightarrow 0$, multiply both sides by $(rt^n)^{|\lambda|}$ and send $r \rightarrow \infty$ using the limit rule (12) to get

$$w_\lambda(x; q, t) = (-1)^{|\lambda|} q^{-|\lambda|-n(\lambda')} t^{n(\lambda)} s^{-|\lambda|} (s)_\lambda \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(t^{j-i+1})_{\lambda_i - \lambda_j}}{(t^{j-i})_{\lambda_i - \lambda_j}} \right\} \\ \cdot \sum_{\mu \subseteq \lambda} \frac{(-1)^{|\mu|} q^{2|\mu|} t^{n(\mu)} q^{n(\mu')}}{(qt^{n-1})_\mu} \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(t^{j-i})_{\mu_i - \mu_j} (qt^{j-i})_{\mu_i - \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j} (t^{j-i+1})_{\mu_i - \mu_j}} \right\} \\ \cdot \left(\lim_{r \rightarrow \infty} \lim_{a \rightarrow 0} W_\mu(q^\lambda t^{\delta(n)}; q, t, art^{1-2n}, ars^{-1}t^{-n}) \right) \cdot w_\mu(xs; q, t) s^{-|\mu|} \quad (42)$$

Simplify this in the one variable case $z = xt^{\delta(n)}$ using (21), and send $s \rightarrow \infty$ to get

$$(x; q^{-1}, t^{-1})_\lambda = \sum_{\mu \subseteq \lambda} u(\lambda, \mu) x^{|\mu|} \quad (43)$$

with the definition of $u(\lambda, \mu)$ above. Similarly, apply the shifts $a \rightarrow as^2$, $b \rightarrow bs$ and $x \rightarrow x/s$ in (41) at the beginning, and follow the same steps

except send s to 0 to get

$$x^{|\lambda|} = \sum_{\mu \subseteq \lambda} v(\lambda, \mu) (x; q^{-1}, t^{-1})_{\mu} \quad (44)$$

where $v(\lambda, \mu)$ is defined as in the theorem. It is clear, by a change of basis argument, that

$$\delta_{\nu\lambda} = \sum_{\mu \subseteq \lambda \subseteq \nu} u(\nu, \lambda) v(\lambda, \mu) \quad (45)$$

Note that the left hand side (44) does not depend on q or t . Now, flip the parameters $q \rightarrow 1/q, t \rightarrow 1/t$, take limit $q \rightarrow 1$, and multiply and divide the summand by $\prod_{i=1}^n 1/(1 - qt^{n-i})^{\mu_i}$ to get

$$x^{|\lambda|} = \sum_{\mu \subseteq \lambda} \prod_{i=1}^n (1 - qt^{n-i})^{\mu_i} \lim_{q \rightarrow 1} v(\lambda, \mu, 1/q, 1/t) \prod_{i=1}^{\mu_1} \langle \bar{x}^i \rangle \quad (46)$$

Multiply and divide the summand in (43) by $t^{(n-1)|\mu|}$, substitute (46) into (43) for $(xt^{n-1})^{|\mu|}$, multiply both sides of this latter identity by $\prod_{i=1}^n 1/(1 - qt^{n-i})^{\nu_i}$ to get

$$\begin{aligned} \prod_{i=1}^n \left\{ \frac{1}{(1 - qt^{n-i})^{\nu_i}} \right\} (x; q^{-1}, t^{-1})_{\nu} &= \sum_{\mu \subseteq \nu} \left(\prod_{i=1}^n \frac{(1 - qt^{n-i})^{\mu_i}}{(1 - qt^{n-i})^{\nu_i}} \right. \\ &\quad \cdot \sum_{\mu \subseteq \lambda \subseteq \nu} u(\nu, \lambda) t^{-(n-1)|\lambda|} \left(\lim_{q \rightarrow 1} v(\lambda, \mu, 1/q, 1/t) \right) \left. \prod_{i=1}^{\mu_1} \langle \bar{x}^i \rangle \right) \quad (47) \end{aligned}$$

Multiplying and dividing the summand now by $q^{n(\nu')} t^{-2n(\mu) + (n-1)|\mu|}$ gives

$$\langle x \rangle_{\nu} = \sum_{\mu \subseteq \nu} q^{-n(\nu')} t^{2n(\mu) - (n-1)|\mu|} s_1(\nu, \mu) \cdot \prod_{i=1}^n \frac{(1 - xt^{n-i})^{\mu_i}}{(1 - qt^{n-i})^{\mu_i}} \quad (48)$$

where $s_1(\nu, \mu)$ is as defined in the theorem. A similar sequence of calculations show that

$$\begin{aligned} \prod_{i=1}^n \frac{(1 - xt^{n-i})^{\nu_i}}{(1 - qt^{n-i})^{\nu_i}} &= \sum_{\mu \subseteq \nu} \left(\prod_{i=1}^n \frac{(1 - qt^{n-i})^{\mu_i}}{(1 - qt^{n-i})^{\nu_i}} \right. \\ &\quad \cdot \sum_{\mu \subseteq \lambda \subseteq \nu} \left(\lim_{q \rightarrow 1} u(\nu, \lambda, 1/q, 1/t) \right) t^{(n-1)|\lambda|} v(\lambda, \mu, q, t) \left. \right) \langle x \rangle_{\mu} \quad (49) \end{aligned}$$

Multiplying and dividing the summand now by $q^{-n(\mu')}t^{2n(\nu)+(1-n)|\nu|}$ gives the explicit formula for the qt -Stirling numbers of the second kind

$$\prod_{i=1}^n \frac{(1 - xt^{n-i})^{\nu_i}}{(1 - qt^{n-i})^{\nu_i}} = \sum_{\mu \subseteq \nu} q^{n(\mu')} t^{-2n(\nu)+(n-1)|\nu|} s_2(\nu, \mu) \cdot \langle x \rangle_\mu \quad (50)$$

Finally, substituting $x \rightarrow q^x$ completes the proof. We conclude by simplifying the flips and limits that entered the formulas above.

It follows immediately from (18) that, if

$$h(\mu, q, t) := \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j}} \right\} \text{ and } g(\mu, q, t) := (qt^{n-1})_\mu \quad (51)$$

then

$$h(\mu, 1/q, 1/t) = t^{2n(\mu) - (n-1)|\mu|} h(\mu, q, t) \quad (52)$$

and

$$g(\mu, 1/q, 1/t) = (-1)^{|\mu|} q^{-|\mu| - n(\mu')} t^{n(\mu) - (n-1)|\mu|} g(\mu, q, t) \quad (53)$$

Thus, flipping the parameters give

$$u(\lambda, \mu, 1/q, 1/t) = \frac{(-1)^{|\mu|} q^{|\mu| + n(\mu')} t^{n(\mu) - (n-1)|\mu|}}{(qt^{n-1})_\mu} \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j}} \right\} w_\mu(q^\lambda t^{\delta(n)}; q, t) \quad (54)$$

and

$$\begin{aligned} v(\lambda, \mu, 1/q, 1/t) &= \frac{t^{(n-1)|\mu|}}{(qt^{n-1})_\mu} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j}} \right\} \cdot w_\mu(q^{-\lambda} t^{-\delta(n)}; 1/q, 1/t) \end{aligned} \quad (55)$$

Multiply and divide both by $(1 - q)^{\mu_1}$, and pass the limit as $q \rightarrow 1$ to get

$$\begin{aligned} &\lim_{q \rightarrow 1} u(\lambda, \mu, 1/q, 1/t) \\ &= (-1)^{|\mu|} t^{n(\mu) - (n-1)|\mu|} \prod_{i=1}^{n-1} \frac{(t)_{\mu_i - \mu_{i+1}}}{(t^{n-i})_{\mu_i}} \prod_{\substack{1 \leq i < j \leq n \\ j \neq i+1}} \left\{ \frac{(t^{j-i})_{\mu_i - \mu_j}}{(t^{j-i-1})_{\mu_i - \mu_j}} \right\} \\ &\quad \cdot \lim_{q \rightarrow 1} \left((1 - q)^{\mu_1} \frac{1}{(q)_{\mu_n}} \prod_{i=1}^{n-1} \frac{1}{(q)_{\mu_i - \mu_{i+1}}} \right) \bar{w}_\mu(q^\lambda t^{\delta(n)}; 1, t) \end{aligned} \quad (56)$$

where \bar{w}_μ is defined as above. Using the limit rule (12), direct calculations give that

$$\lim_{q \rightarrow 1} \left((1-q)^{\mu_1} \frac{1}{(q)_{\mu_n}} \prod_{i=1}^{n-1} \frac{1}{(q)_{\mu_i - \mu_{i+1}}} \right) = \mu_n! \prod_{i=1}^{n-1} (\mu_i - \mu_{i+1})! \quad (57)$$

Hence

$$\lim_{q \rightarrow 1} u(\lambda, \mu, 1/q, 1/t) = (-1)^{|\mu|} t^{n(\mu) - (n-1)|\mu|} \bar{w}_\mu(q^\lambda t^{\delta(n)}; 1, t) f(\mu, q, t) \quad (58)$$

Similarly,

$$\lim_{q \rightarrow 1} v(\lambda, \mu, 1/q, 1/t) = t^{(n-1)|\mu|} \bar{w}_\mu(q^{-\lambda} t^{-\delta(n)}; 1, 1/t) f(\mu, q, t) \quad (59)$$

which completes the proof.

Remark 2. The immediate properties of the Stirling numbers established in [16] are listed as follows:

(a) The multiple Stirling numbers $s_1(\nu, \mu)$ and $s_2(\nu, \mu)$ admit explicit combinatorial formulas which are derived in the Theorem above.

(b) These explicit formulas reduce to those for the q -Stirling numbers given by Kim in [29] for $n = 1$. Moreover, sending $q \rightarrow 1$ in that case yields classical Stirling numbers of both types.

(c) The matrix m with entries $m_{\lambda\mu} = s_1(\lambda, \mu)$ is invertible in the sense of V algebra, and its inverse is given by $m_{\lambda\mu}^{-1} = s_2(\lambda, \mu)$. More precisely, we have

$$\delta_{\nu\lambda} = \sum_{\mu \subseteq \lambda \subseteq \nu} s_1(\nu, \lambda) s_2(\lambda, \mu) = \sum_{\mu \subseteq \lambda \subseteq \nu} s_2(\nu, \lambda) s_1(\lambda, \mu) \quad (60)$$

which follows immediately from the inversion relation (45).

(d) Similar to the one dimensional case for the q -Stirling numbers, we have

$$s_1(\lambda, \lambda) = s_2(\lambda, \lambda) = 1 \quad (61)$$

for an arbitrary n -part partition λ .

(e) Setting $t = q^\alpha$ and sending $q \rightarrow 1$ gives the multiple ordinary α -Stirling numbers of the first and second kind.

4. Additional multiple Stirling number identities

We derive some additional new properties of the multiple Stirling numbers in this section.

(1) First note that $\lim_{x \rightarrow 0} \langle x \rangle_\mu = \prod_{i=1}^n 1/(1 - qt^{n-i})^{\mu_i}$ follows readily from the formula (31) written in the qt -angle brackets $\langle \bar{x} \rangle_\mu$ as

$$\langle \bar{x} \rangle_\mu = \prod_{i=1}^n \left\{ \frac{1}{(1 - qt^{n-i})^{\mu_i}} \right\} (x; q^{-1}, t^{-1})_\mu = \prod_{i=1}^n \left\{ \frac{(xt^{i-1}; q^{-1})_{\mu_i}}{(1 - qt^{n-i})^{\mu_i}} \right\} \quad (62)$$

In the multiple case, setting $x = 0$ in (48) and (50) respectively gives

$$\prod_{i=1}^n \frac{1}{(1 - qt^{n-i})^{\nu_i}} = \sum_{\mu \subseteq \nu} q^{-n(\nu')} t^{2n(\mu) + (n-1)|\mu|} s_1(\lambda, \mu) \prod_{i=1}^n \frac{1}{(1 - qt^{n-i})^{\mu_i}} \quad (63)$$

and

$$\prod_{i=1}^n \frac{1}{(1 - qt^{n-i})^{\nu_i}} = \sum_{\mu \subseteq \nu} q^{n(\mu')} t^{-2n(\nu) + (n-1)|\nu|} s_2(\nu, \mu) \prod_{i=1}^n \frac{1}{(1 - qt^{n-i})^{\mu_i}} \quad (64)$$

These appear to be new identities, even in the one dimensional case $n = 1$.

(2) Note that, for an n -part partition ν with $\nu_n \neq 0$, the bracket $\langle x \rangle_\nu$ has roots at $x = t^{1-j} q^{m_j}$ for $j = 1, \dots, n$, and $m_j = 0, \dots, \nu_j - 1$. The limit bracket $\prod_{i=1}^n (1 - xt^{n-i})^{\mu_i} / (1 - qt^{n-i})^{\mu_i}$ has roots at $x = t^{1-j}$ for $j = 1, \dots, n$. Therefore, if we set $x = t^{1-j} q^{m_j}$ (for some $m_j < \nu_j$) in (50) we get

$$0 = \sum_{\mu \subsetneq \nu} q^{-n(\nu')} t^{2n(\mu) - (n-1)|\mu|} \prod_{i=1}^n \frac{(1 - q^{m_j} t^{1-j+n-i})^{\mu_i}}{(1 - qt^{n-i})^{\mu_i}} s_1(\nu, \mu) \quad (65)$$

where the summation is over all partitions $\mu \subsetneq \nu$, that is all partitions $\mu \subseteq \nu$ such that $\mu_j \leq m_j$. This inequality follows from the vanishing property of the w functions (22).

In the particular case, setting $x = q$ in (48) gives

$$0 = \sum_{\mu \subseteq \nu} q^{-n(\nu')} t^{2n(\mu) - (n-1)|\mu|} s_1(\lambda, \mu) \quad (66)$$

which is an analogue of $\sum_{k=0}^n s_1(n, k) = 0$ in the classical case.

In another special case, setting $x = t^{1-j}$ (i.e., $m_j = 0$) in (48) and (50) would amount to vanishing of all brackets $\langle x \rangle_\mu$ except the ones in whose index the j -th part (thus all parts $j+1, \dots, n$ after j) are 0. That is, the brackets will be nonzero only for partitions such as $\mu = (\mu_1, \dots, \mu_{j-1}, 0, \dots, 0)$, and others will vanish. This is particularly interesting, for the substitution $x = t^{1-j}$ the limit brackets $\prod_{i=1}^n (1 - xt^{n-i})^{\mu_i} / (1 - qt^{n-i})^{\mu_i}$ also vanish except for partitions $\mu = (\mu_1, \dots, \mu_{n-j}, 0, \dots, 0)$.

In general, setting $x = t^{1-j}$ in (48) and (50) respectively gives

$$0 = \sum_{\mu \subsetneq \nu} q^{-n(\nu')} t^{2n(\mu) - (n-1)|\mu|} s_1(\nu, \mu) \prod_{i=1}^n \frac{(1 - t^{1-j+n-i})^{\mu_i}}{(1 - qt^{n-i})^{\mu_i}} \quad (67)$$

where the sum is over all partitions $\mu \subsetneq \nu$ such that $\mu = (\mu_1, \dots, \mu_{j-1}, 0, \dots, 0)$. Likewise,

$$0 = \sum_{\mu \subsetneq \nu} q^{n(\mu')} t^{-2n(\nu) + (n-1)|\nu|} s_2(\nu, \mu) \langle t^{1-j} \rangle_\mu \quad (68)$$

where the sum is over all partitions $\mu \subsetneq \nu$ such that $\mu = (\mu_1, \dots, \mu_{n-j}, 0, \dots, 0)$. The particular cases when $j = 1$ in (67) and $j = n$ in (68) show that the multiple Stirling numbers vanish when $\mu = \bar{0} = (0, \dots, 0) \in \mathbb{C}^n$ as in the classical case. That is,

$$s_1(\lambda, \bar{0}) = s_2(\lambda, \bar{0}) = 0$$

for any n -part partition λ with $\lambda_n \neq 0$.

(3) Recall that, $s_1(n, m) = -s_2(n, m) = -\binom{n}{2}$ when $n-m = 1$ for the classical Stirling numbers. Similarly, if the index partitions satisfy $|\lambda| - |\tilde{\lambda}| = 1$, we have that

$$s_1(\lambda, \tilde{\lambda}) = -s_2(\lambda, \tilde{\lambda})$$

exactly as in the one dimensional case.

The proof follows easily from the inversion (60) relation, and the observation that there are only two partitions between λ and $\tilde{\lambda}$ under the inclusion ordering, namely the two partitions themselves. That is,

$$0 = \delta_{\lambda\tilde{\lambda}} = \sum_{\tilde{\lambda} \subseteq \mu \subseteq \lambda} s_1(\lambda, \mu) s_2(\mu, \tilde{\lambda}) \quad (69)$$

which implies that $s_1(\lambda, \lambda) s_2(\lambda, \tilde{\lambda}) = -s_1(\lambda, \tilde{\lambda}) s_2(\tilde{\lambda}, \tilde{\lambda})$. That the diagonal entries of both type of multiple qt -Stirling numbers are 1 by (61) is now enough to conclude.

5. Conclusion

We have derived several interesting summation identities for the multiple qt -Stirling numbers in the present paper. We will write down additional properties such as the recurrence relations they satisfy, their explicit evaluations in certain special cases, their combinatorial interpretation, generating functions, and connections to other families of multiple special numbers in an upcoming article. The Stirling numbers have interesting connections to various branches in mathematics such as the one expressed in the classical Dobinski's formula. Such relations will also be formulated in that paper.

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